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TIME HARMONIC ACOUSTIC RADIATION FROM A SUBMERGED ELASTIC SHELL--ETC(U)
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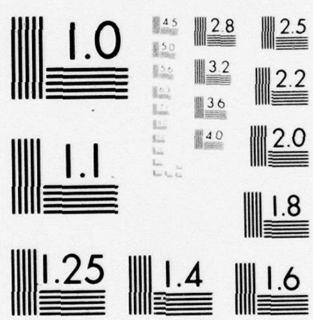

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State University of New York at Buffalo

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Report No. 97

TIME HARMONIC ACOUSTIC RADIATION FROM A SUBMERGED ELASTIC SHELL DEFINED BY NONCONCENTRIC CYLINDERS

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by

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ABSTRACT

The acoustic radiation from an elastic cylindrical shell defined by nonconcentric cylindrical surfaces and submerged in an acoustic fluid is studied. The driving mechanism is a time harmonic internal pressure. In particular, the influence of the nonconcentricity parameter, δ, on the resulting radiation field is examined. The solution is obtained by means of an expansion in powers of this parameter about the concentric solution, $\delta = 0$. Each order of solution is solved in closed form by means of boundary integral equation methods. Results are given for the zeroth and first order solutions and it is seen that a j^{th} driving mode will introduce nearest neighbor ((j-1) and (j+1)) modes in the first order solution.

I. INTRODUCTION

We consider the radiation into an infinite acoustic fluid from a submerged elastic shell subjected to a time harmonic internal pressure with some prescribed spatial distribution. The shell is bounded by surfaces of separable geometry, i.e., such that the governing (Helmholtz) partial differential equations have modal solutions, that are nonconcentric, i.e., the origin of the coordinate system in which the outer boundary is "separable" is displaced from that for the inner boundary by a nonconcentricity distance, δ . If δ were zero, the entire problem could be solved by separation of variables but with δ nonzero, the problem is not separable.

The purpose of this investigation is to study the influence of the nonconcentricity on the resulting radiation field by making an expansion of the solution in powers of δ . Clearly the zeroth order will give the classical separation of variables solution. Higher order solutions will then represent the influence of nonconcentricity on this basic solution.

Rather than work with the governing differential equations, we convert to a boundary integral equation representation which eventually allows us to solve a set of algebraic equations for each mode in each order to obtain the exact solution.

To simplify the algebra, the submerged elastic body will be taken as an infinite cylinder in plane strain thereby reducing the problem to a two dimensional case. A specific example of nonconcentric circular boundaries is calculated as an illustration. Analogous three dimensional problems are also being studied - the axisymmetric three dimensional case requires essentially no more effort than this two dimensional problem.

The full equations of elasticity shall be used rather than approximating set of shell equations since the use of the boundary integral equation method

reduces the problem by one dimension anyway thereby accomplishing one of the main purposes of shell theory.

This type of problem has been studied previously for acoustic radiators, both by the boundary integral equation method [1] and by shifting and superposition theorems [2].

II. GENERAL FORMULATION

A. ACOUSTIC FLUID EQUATIONS

The infinite acoustic fluid exterior to the elastic body is described in terms of an acoustic (excess) pressure field, p, which satisfies a Helmholtz equation

$$\nabla^2 p + k^2 p = 0 \tag{1}$$

where $k = \omega/c$ is the acoustic wave number, ω is the frequency of the time harmonic dependence (taken in the form $\exp[-i\omega t]$) and c is the acoustic sound speed. This equation is readily replaced by a Helmholtz (or Weber, for the two dimensional case) integral equation; e.g., [3]

$$\varepsilon p(\bar{r}) = \oint_\Gamma \left[-p(r_0) \frac{\partial G(\bar{r}, \bar{r}_0)}{\partial n_0} + G(\bar{r}, \bar{r}_0) \frac{\partial p(\bar{r}_0)}{\partial n_0} \right] dS_0 \tag{2}$$

where $G = \frac{i}{4} H_0^{(1)}(kR)$ in two dimensions and $G = \frac{1}{4} \exp[ikR]/R$ in three dimensions. R is the distance between the field point $\bar{r}$ and the integration variable, $\bar{r}_0$, n_0 is the outward normal from the fluid, Γ is the line or surface bounding the fluid, i.e., the elastic solid-acoustic fluid interface with length or area element dS, and $\varepsilon = 0, 1/2, 1$ depending on whether the field point is exterior to the fluid, on the interface or interior to the fluid. The radiation condition is automatically satisfied by the choice of G. Since only Hankel functions of the first kind will occur, the superscript (1) will be dropped.

The fluid velocity field is related to the pressure field through the acoustic velocity potential ϕ, which could also have been used as the dependent variable

$$\text{(3)} \quad \begin{aligned} &\text{(a)} \quad \bar{v} = -\nabla \phi \\ &\text{(b)} \quad p = \rho_f \, \partial\phi/\partial t = -i\omega \rho_f \phi \end{aligned}$$

where ρ_f is the acoustic fluid density.

B. ELASTIC SHELL EQUATIONS

The elastic shell involves a finite domain with an inner and an outer boundary (treating the infinite cylinder as a two dimensional plane strain problem). It is best described by the displacement potentials Φ and $\bar{\Psi}$ defined by the displacement field $\bar{u}$ as;

$$(4) \qquad \bar{u} = \nabla \Phi + \nabla \times \bar{\Psi} \quad ; \quad \nabla \cdot \bar{\Psi} = 0$$

These potentials also satisfy Helmholtz equations

$$(5) \quad \begin{array}{ll} \text{(a)} & \nabla^2 \Phi + k_D^2 \Phi = 0 \\ \text{(b)} & \nabla^2 \bar{\Psi} + k_R^2 \Psi = 0 \end{array}$$

where $k_D = \omega/c_D$, $k_R = \omega/c_R$ and $c_D = \sqrt{(\lambda+2\mu)/\rho_s}$ is the dilatational wave speed, $c_R = \sqrt{\mu/\rho_s}$ is the rotational wave speed in an elastic medium of density ρ_s and Lame parameters, λ and μ, e.g., [4].

In the two dimensional plane strain case considered here, the vector displacement potential $\bar{\Psi}$ has only one nonzero component and will be written as a scalar, ψ.

The displacement potentials are also related to the dilatation Δ and the rotation $\bar{\omega}_R$

$$(6) \quad \begin{array}{ll} \text{(a)} & \Delta = - k_D^2 \Phi \\ \text{(b)} & \omega_R = + \frac{1}{2} k_R^2 \Psi \end{array}$$

which may also be used as dependent variables.

Clearly these displacement potentials also satisfy Helmholtz integral equations analogous to equation (2) with two major changes. The wavenumber k must be changed to either k_D or k_R and the surface Γ now consists of two parts, the inner and the outer boundaries. We note that the direction of n_o changes

sign between these boundaries since it is required to be "outward" from the elastic medium. Then

$$\text{(7)} \quad \varepsilon\,\Phi(\bar{r}) = \oint_{\Gamma_o}\left[-\Phi(\bar{r}_o)\frac{\partial G_D}{\partial n_o} + G_D\frac{\partial \Phi(\bar{r}_o)}{\partial n_o}\right] dS_{OUTER} + \oint_{\Gamma_i}\left[-\Phi(\bar{r}_o)\frac{\partial G_D}{\partial n_o} + G_D\frac{\partial \Phi(\bar{r}_o)}{\partial n_o}\right] dS_{INNER}$$

$$\text{(8)} \quad \varepsilon\,\Psi(\bar{r}) = \oint_{\Gamma_o}\left[-\Psi(\bar{r}_o)\frac{\partial G_R}{\partial n_o} + G_R\frac{\partial \Psi(\bar{r}_o)}{\partial n_o}\right] dS_{OUTER} + \oint_{\Gamma_i}\left[-\Psi(\bar{r}_o)\frac{\partial G_R}{\partial n_o} + G_R\frac{\partial \Psi(\bar{r}_o)}{\partial n_o}\right] dS_{INNER}$$

where G_D, G_R are equivalent to the previous G with k_D, k_R replacing k .

$\bar{r}$ in any of these boundary integral equations will be placed on (or infinitesimally near to) either the inner or the outer surface. Thus each of the above boundary integral equations will produce two separate integral equations.

C. BOUNDARY CONDITIONS

We supplement these equations with five boundary conditions requiring zero shear stress, continuity of the normal displacement or velocity field and continuity of the normal stress field at the outer boundary of the elastic shell and zero shear stress and continuity of normal stress at the inner boundary. (Actually, those at the outer boundary might better be called interface conditions.)

For plane strain in a two dimensional orthogonal curvilinear coordinate system (α,β), we have the infinitesimal strains, e.g., [5]

(9)

(a) $$\epsilon_{\alpha\alpha} = (1/h_1)\,\partial u_\alpha/\partial\alpha + (1/h_1 h_2)\,u_\beta\,\partial h_1/\partial\beta$$

(b) $$\epsilon_{\beta\beta} = (1/h_2)\,\partial u_\beta/\partial\beta + (1/h_1 h_2)\,u_\alpha\,\partial h_2/\partial\alpha$$

(c) $$\epsilon_{\alpha\beta} = (h_2/h_1)\,\partial[u_\beta/h_2]/\partial\alpha + (h_1/h_2)\,\partial[u_\alpha/h_1]/\partial\beta$$

and the dilatation and rotation

(10)

(a) $$\Delta = [\,\partial(u_\alpha h_2)/\partial\alpha + \partial(u_\beta h_1)/\partial\beta\,]\,/\,[h_1 h_2]$$

(b) $$\omega_R = [\,\partial(u_\beta h_2)/\partial\alpha - \partial(u_\alpha h_1)/\partial\beta\,]\,/\,[2 h_1 h_2]$$

where h_1 and h_2 are the metric coefficients (scale factors) for these coordinates, $(h_3 = 1)$.

We can also write out equation (4) specifically:

(11)

(a) $$u_\alpha = (1/h_1)\,\partial\Phi/\partial\alpha + (1/h_2)\,\partial\Psi/\partial\beta$$

(b) $$u_\beta = (1/h_2)\,\partial\Phi/\partial\beta - (1/h_1)\,\partial\Psi/\partial\alpha$$

We shall choose the boundaries to be surfaces of constant α ; the α direction is then normal to these surfaces. Continuity of normal stress requires

$$-p = \sigma_{\alpha\alpha} = \lambda\Delta + 2\mu\epsilon_{\alpha\alpha}$$

which reduced to

(12) $$-p = -\rho_S\omega^2\Phi + 2\mu\left\{\frac{-1}{h_1^2h_2}\frac{\partial h_2}{\partial\alpha}\frac{\partial\Phi}{\partial\alpha} + \frac{1}{h_2^3}\frac{\partial h_2}{\partial\beta}\frac{\partial\Phi}{\partial\beta}\right.$$

$$\left. - \frac{1}{h_2^2}\frac{\partial^2\Phi}{\partial\beta^2} - \frac{1}{h_1h_2^2}\frac{\partial h_2}{\partial\alpha}\frac{\partial\Psi}{\partial\beta} - \frac{1}{h_1^2h_2}\frac{\partial h_1}{\partial\beta}\frac{\partial\Psi}{\partial\alpha} + \frac{1}{h_1h_2}\frac{\partial^2\Psi}{\partial\alpha\partial\beta}\right\}$$

Similarly, vanishing of the shear stress requires

(13) $$0 = \frac{2}{h_1h_2}\frac{\partial^2\Phi}{\partial\alpha\partial\beta} - \frac{2}{h_1^2h_2}\frac{\partial h_1}{\partial\beta}\frac{\partial\Phi}{\partial\alpha} - \frac{2}{h_1h_2^2}\frac{\partial h_2}{\partial\alpha}\frac{\partial\Phi}{\partial\beta}$$

$$+ \frac{2}{h_1^2h_2}\frac{\partial h_2}{\partial\alpha}\frac{\partial\Psi}{\partial\alpha} - \frac{2}{h_2^3}\frac{\partial h_2}{\partial\beta}\frac{\partial\Psi}{\partial\beta} + \frac{2}{h_2^2}\frac{\partial^2\Psi}{\partial\beta^2} + k_R^2\Psi$$

and continuity of normal velocity requires

(14) $$\frac{\partial p}{\partial\alpha} = \rho_F\omega^2\left[\frac{1}{h_1}\frac{\partial\Phi}{\partial\alpha} + \frac{1}{h_2}\frac{\partial\Psi}{\partial\beta}\right]$$

D. SOLUTION PROCEDURE

There will in general be ten equations, two boundary conditions plus two integral equations (on Φ , ψ) evaluated at the inner boundary and three boundary conditions plus three integral equations (on p, Φ and ψ) evaluated at the outer boundary. Treating p at the inner boundary as known, these equations describe the behavior of Φ, $\partial\Phi/\partial\alpha$, ψ , $\partial\psi/\partial\alpha$ on the inner boundary and p, $\partial p/\partial\alpha$, Φ, $\partial\Phi/\partial\alpha$, ψ, $\partial\psi/\partial\alpha$ on the outer boundary as ten unknown functions of the remaining independent variable β . To emphasize the use of a different origin, we shall use the symbols (α,β) for the inner boundary coordinate system and the symbols (α',β') for the outer. We also use a subscript o to represent an integration coordinate. To simplify the algebra as much as possible, the inner boundary forcing pressure will be taken proportional to the jth eigenfunction of the separated Helmholtz equation for that surface. Finally the inner boundary will be a surface of constant $\alpha = a$ and the outer of constant $\alpha' = b$, as shown in Figure 1.

We recognize that the inner boundary variables may be expanded as a series in terms of the inner boundary eigenfunctions; similarly for the outer boundary variables. The ψ and $\partial\psi/\partial\alpha$ series will be in terms of the antisymmetric modes if p, Φ, etc. are in terms of the symmetric modes and vice-versa as required by the boundary conditions. Since each boundary condition involves only a single boundary, they must be separable, i.e., each mode of the series expansion must satisfy the boundary conditions independently of the other modes thereby providing five algebraic equations on the unknown coefficients of each mode of the expansion.

The boundary integral equations, however, have one set of terms wherein the integration and field point lie on the same surface, thereby allowing a direct integration to be carried out independently of δ , and another set of "mixed"

terms where the integration and field point lie on different surfaces, thereby involving δ explicitly.

We can carry out the integration of the first type quite simply since the Greens' function possesses an eigenfunction expansion in terms of the same functions (either completely inner or completely outer boundaries) as did the other dependent variables. These integrals then pose no real difficulty. Integrals of the mixed (second) type would pose no difficulty if δ were zero, i.e., in the concentric cases, the eigenfunctions in terms of β must be identical on surfaces of constant α . This then suggests a power series expansion in terms of δ about δ equal to zero of all of the algebraic coefficients of all of the dependent variable series expansions as well as that of the Greens' function for the mixed integrals. The equations for the boundary conditions and for the first type of integral do not involve δ explicitly and expressions for each order of δ are identical to the original expression. The mixed integrals involve a product of two series in powers of δ and thereby introduce a coupling of orders. The zeroth order solution will, of course, be that obtainable directly by separation of variables for the concentric case. The first order solution is coupled to the zeroth order solution through the mixed integrals -- this in the first order solution will be found to introduce "nearest neighbor modes" to each mode of the zeroth order solution, etc.

By taking the forcing function to excite only a single mode in the zeroth order solution, we find ten algebraic equations to solve for the ten unknown coefficients in the zeroth order series expansion, two sets of ten algebraic equations for the first order nearest neighbor modes, etc.

We shall illustrate this by an example.

III. SPECIFIC EXAMPLE - CIRCULAR CYLINDER

A. BOUNDARY CONDITIONS AND EIGENFUNCTION EXPANSIONS

Let us consider nonconcentric circular boundaries such that $(\alpha,\beta) = (r,\theta)$ and $(\alpha',\beta') = (r',\theta')$ with $r = a$ the inner and $r'=b$ the outer boundaries. The scale factors are $h_1 = 1$ and $h_2 = r$ and the boundary conditions become

$$-p = -\rho_s \omega^2 \Phi + 2\mu\left[\frac{-1}{r}\frac{\partial \Phi}{\partial r} - \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} - \frac{1}{r^2}\frac{\partial \Psi}{\partial \theta} + \frac{1}{r}\frac{\partial^2 \Psi}{\partial r \partial \theta}\right] \tag{15}$$

$$0 = \mu\left[\frac{2}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta} + \frac{2}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial \Phi}{\partial \theta} + \frac{2}{r}\frac{\partial \Psi}{\partial r}\right] + \rho_s \omega^2 \Psi \tag{16}$$

$$\frac{\partial p}{\partial r} = \rho_F \omega^2 \left[\frac{\partial \Phi}{\partial r} + \frac{1}{r}\frac{\partial \Psi}{\partial \theta}\right] \tag{17}$$

We have the appropriate eigenfunction expansions

$$\left[p, \Phi, \frac{\partial \Phi}{\partial r}\right]_{r=a} = \sum_{n=0}^{\infty}\left[A_{1n}, A_{3n}, A_{4n}\right]\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(n\theta) \tag{18}$$

$$\left[\Psi, \frac{\partial \Psi}{\partial r}\right]_{r=a} = \sum_{n=0}^{\infty}\left[A_{5n}, A_{6n}\right]\begin{Bmatrix}\cos\\ -\sin\end{Bmatrix}(n\theta) \tag{19}$$

$$\left[p, \frac{\partial p}{\partial r'}, \Phi, \frac{\partial \Phi}{\partial r'}\right]_{r'=b} = \sum_{n=0}^{\infty}\left[B_{1n}, B_{2n}, B_{3n}, B_{4n}\right]\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(n\theta') \tag{20}$$

$$\left[\Psi, \frac{\partial \Psi}{\partial r'}\right]_{r'=b} = \sum_{n=0}^{\infty}\left[B_{5n}, B_{6n}\right]\begin{Bmatrix}\cos\\ -\sin\end{Bmatrix}(n\theta') \tag{21}$$

which reduce the boundary conditions to algebraic equations

$$-A_{1n} = -\rho_s \omega^2 A_{3n} + 2\mu\left\{-\frac{1}{a}A_{4n} + \frac{n^2}{a^2}A_{3n} + \frac{n}{a^2}A_{5n} - \frac{n}{a}A_{6n}\right\} \tag{22}$$

$$0 = 2\mu\left\{\frac{1}{a}A_{4n} - \frac{n^2}{a^2}A_{5n} - \frac{n}{a^2}A_{3n} + \frac{1}{a}A_{6n}\right\} + \rho_s \omega^2 A_{5n} \tag{23}$$

$$-B_{1n} = -\rho_s \omega^2 B_{3n} + 2\mu\left\{-\frac{1}{b}B_{4n} + \frac{n^2}{b^2}B_{3n} + \frac{n}{b^2}B_{5n} - \frac{n}{b}B_{6n}\right\} \tag{24}$$

$$0 = 2\mu\left\{\frac{1}{b}B_{4n} - \frac{n^2}{b^2}B_{5n} - \frac{n}{b^2}B_{3n} + \frac{1}{b}B_{6n}\right\} + \rho_s \omega^2 B_{5n} \tag{25}$$

$$B_{2n} = \rho_F \omega^2\left[B_{4n} - \frac{n}{b}B_{5n}\right] \tag{26}$$

We now consider the eigenfunction expansion of the Greens' function:

$$G = \frac{i}{4} H_0(kR) = \frac{i}{4} \sum_{m=0}^{\infty} \epsilon_m \cos[m(\theta-\theta_0)]\, J_m(k\alpha)\, H_m(k\beta) \tag{27}$$

where $\varepsilon_o = 1$, $\varepsilon_m = 2$ $(m \neq 0)$, $R^2 = \alpha^2 + \beta^2 - 2\alpha\beta\cos(\theta-\theta_o)$ and β must be greater than α. [These β,α have no relationship to the general coordinates used before but are merely convenient labels here for the field and integration points.] θ or θ_o may be primed or unprimed depending on the particular cases and α,β may be either a or b . To keep α from being identically equal to β , we choose field points just "off" the integration surface. In all cases, we shall choose them to be "outside" of the region over which the original integration is performed, i.e., for the infinite acoustic fluid we choose r' at b^- while for the elastic solid we choose r' at b^+ and r at a^- for field points at the outer and inner boundaries respectively.

B. BOUNDARY INTEGRAL EQUATION - FLUID

The boundary integral equation for the infinite acoustic fluid becomes

$$(28)\quad 0 = \int_0^{2\pi} \Bigg[\Bigg\{ \sum_{n=0}^{\infty} - B_{2n} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (n\theta_o') \cdot \sum_{m=0}^{\infty} \epsilon_m \cos[m(\theta' - \theta_o')] J_m(kr') H_m(kr_o') \Bigg\}$$

$$+ \Bigg\{ \sum_{n=0}^{\infty} B_{1n} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (n\theta_o') \cdot \sum_{m=0}^{\infty} k\, \epsilon_m \cdot \cos[m(\theta' - \theta_o')] J_m(kr') H_m'(kr_o') \Bigg\} \Bigg]_{\substack{r' = b^-, \\ r_o' = b}} b\, d\theta_o'$$

where we have identified α with $r'=b^-$ as smaller in magnitude than β which is identified with $r_o'=b$. This tells us essentially which term to differentiate in the Greens' function series, and leads to a simple relationship

$$(29)\quad 0 = \sum_{n=0}^{\infty} \Big\{ B_{2n} J_n(kb) H_n(kb) - k B_{1n} J_n(kb) H_n'(kb) \Big\} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (n\theta')$$

which by the orthogonality of the eigenfunctions reduces, for $J_n(kb) \neq 0$, to

$$(30)\quad B_{2n} = B_{1n} \cdot k \cdot H_n'(kb) / H_n(kb)$$

C. BOUNDARY INTEGRAL EQUATION - SOLID

W may now consider the boundary integral equations for the elastic shell. First, we examine the outer boundary equations with $r'=b^+$, i.e., we place the field point just outside of the outer boundary of the elastic body. Then, for example, we have

$$(31) \quad 0 = \int_0^{2\pi} \left\{ -\Phi(b) \frac{\partial G_D}{\partial r_o'} + G_D \frac{\partial \Phi(b)}{\partial r_o'} \right\}_{r_o'=b,\, r'=b^+} b\, d\theta_o'$$
$$+ \int_0^{2\pi} \left\{ +\Phi(a) \frac{\partial G_D}{\partial r_o} - G_D \frac{\partial \Phi(a)}{\partial r_o} \right\}_{r_o=a,\, r'=b} a\, d\theta_o$$

The first integral leads directly to terms involving $\left\{ {\sin \atop \cos} \right\} n\theta'$ as in the acoustic fluid case above. The second integral does not. We cannot mix θ and θ' in equation (31) as long as they have different origins. To avoid this dilemma, we expand about $\delta=0$ such that, for example,

$$(32) \quad G = \frac{i}{4} H_o(kR) = \sum_{\ell=0}^{\infty} \delta^\ell G^{(\ell)}[\delta=0] = \frac{i}{4} \sum_{\ell=0}^{\infty} \delta^\ell \left[H_o(kR)\right]^{(\ell)}_{\delta=0}$$

The δ terms may be removed from the integration leaving an integral over $G_D^{(\ell)}(\delta=0)$, i.e., some Greens function with θ and θ' measured from the same origin.

Since this is simply a Taylor Series in δ, we have (see reference [ ] for details)

$$(33) \quad G^{(\ell)}[\delta=0] = \frac{1}{\ell!} \left[\frac{\partial^\ell H_o(kR)}{\partial \delta^\ell} \right]_{\delta=0}$$

We cannot use the eigenfunction, equation (27), expansion of G directly since it applies only when θ and θ' are measured from the same origin. Instead we can use, for example,

$$(34) \quad \left[H_o(kR)\right]^{(1)}_{\delta=0} = \left[\frac{\partial H_o(kR)}{\partial \delta}\right]_{\delta=0} = -k\left[H_1(kR) \frac{\partial R}{\partial \delta}\right]_{\delta=0}$$

where

$$(35) \quad R^2 = \lambda + \lambda_o^2 - 2\lambda\lambda_o \cos(\Theta_o - \Theta) + 2\delta[\lambda_o \cos\Theta_o - \lambda\cos\Theta] + \delta^2$$

and

$$(36) \quad \frac{\partial R}{\partial \delta} = [\delta + \lambda_o \cos\Theta_o - \lambda\cos\Theta]/R$$

Here λ and λ_o are distances measured along θ and θ_o respectively from origins which are a distance δ apart on the x axis. We then have the first order kernel

$$(37) \quad [H_o(kR)]^{(1)}_{\delta=0} = -k\left[H_1(kR)\cdot[\lambda_o\cos\Theta_o - \lambda\cos\Theta]/R\right]_{\delta=0}$$

For $\lambda = r$, $\lambda_o = r_o'$ we have

$$(38) \quad \int_0^{2\pi} \begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(n\Theta_o')\,[H_o(kR)]^{(1)}_{\delta=0}\,d\Theta_o'$$

$$= \pi k H_n(kr_o')\left[\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}((n+1)\Theta)\,J_{n+1}(kr) - \begin{Bmatrix}\sin\\ \cos\end{Bmatrix}((n-1)\Theta)\,J_{n-1}(kr)\right]$$

We may set r_o' equal to b and r equal to a after any appropriate differentiations are carried out.

Similarly for $\lambda = r'$ and $\lambda_o = r_o$

$$(39) \quad \int_0^{2\pi} \begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(n\Theta_o)\,[H_o(kR)]^{(1)}_{\delta=0}\,d\Theta_o$$

$$= \pi k J_n(kr_o)\left[\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}((n-1)\Theta')\cdot H_{n-1}(kr') - \begin{Bmatrix}\sin\\ \cos\end{Bmatrix}((n+1)\Theta')\,H_{n+1}(kr')\right]$$

Clearly we must also expand all of the other dependent variables in powers of δ - or equivalently the coefficients of their eigenfunction expansions:

$$(40) \quad \begin{aligned} &(a) \quad A_{jm} = \sum_{\ell=0}^{\infty} \delta^\ell A_{jm}^{(\ell)} \quad ; \quad j = 1,3,4,5,6 \\ &(b) \quad B_{jm} = \sum_{\ell=0}^{\infty} \delta^\ell B_{jm}^{(\ell)} \quad ; \quad j = 1,2,3,4,5,6 \end{aligned}$$

If we examine the boundary conditions, equations (15-17) and the acoustic fluid boundary integral equation, (20), we see that they must hold for each separate order (power of δ) since δ does not appear explicitly. The elastic boundary integral equation (31) however becomes

$$
\begin{aligned}
(41)\quad 0 = & \int_0^{2\pi} \Bigg\{ -\sum_{l=0}^{\infty} \delta^l \sum_{n=0}^{\infty} B_{3n}^{(l)} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (n\theta_0') \frac{\partial G_D}{\partial r_0'} \\
& \qquad + \sum_{l=0}^{\infty} \delta^l \sum_{n=0}^{\infty} B_{4n}^{(l)} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (n\theta_0') \, G_D \Bigg\}_{r_0'=b,\, r'=b^+} b\, d\theta_0' \\
& + \int_0^{2\pi} \Bigg\{ \sum_{l=0}^{\infty} \delta^l \sum_{n=0}^{\infty} A_{3n}^{(l)} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (n\theta_0) \cdot \frac{\partial}{\partial r_0} \sum_{i=0}^{\infty} \delta^i G_D^{(i)} \\
& \qquad - \sum_{l=0}^{\infty} \delta^l \sum_{n=0}^{\infty} A_{4n}^{(l)} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (n\theta_0) \sum_{i=0}^{\infty} \delta^i G_D^{(i)} \Bigg\}_{r_0=a,\, r'=b^+} a\, d\theta_0
\end{aligned}
$$

which may be expanded in powers of δ as

$$
\begin{aligned}
(42)\quad 0 = & \int_0^{2\pi} \Bigg\{ \sum_{n=0}^{\infty} \left[-B_{3n}^{(0)} \frac{\partial G_D}{\partial r_0'} + B_{4n}^{(0)} G_D \right] \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (n\theta_0') \Bigg\}_{r_0'=b,\, r'=b^+} b\, d\theta_0' \\
& + \int_0^{2\pi} \Bigg\{ \sum_{n=0}^{\infty} \left[+A_{3n}^{(0)} \frac{\partial G_D^{(0)}}{\partial r_0} - A_{4n}^{(0)} G_D^{(0)} \right] \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (n\theta_0) \Bigg\}_{r_0=a,\, r'=b^+} a\, d\theta_0 \\
& + \delta \Bigg[\int_0^{2\pi} \Bigg\{ \sum_{n=0}^{\infty} \left[-B_{3n}^{(1)} \frac{\partial G_D}{\partial r_0'} + B_{4n}^{(1)} G_D \right] \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (n\theta_0') \Bigg\}_{r_0'=b,\, r'=b^+} b\, d\theta_0' \\
& \quad + \int_0^{2\pi} \Bigg\{ \sum_{n=0}^{\infty} \left[A_{3n}^{(1)} \frac{\partial G_D^{(0)}}{\partial r_0} - A_{4n}^{(1)} G_D^{(0)} + A_{3n}^{(0)} \frac{\partial G_D^{(1)}}{\partial r_0} - A_{4n}^{(0)} G_D^{(1)} \right] \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (n\theta_0) \Bigg\}_{r_0=a,\, r'=} \\
& \qquad \cdot a\, d\theta_0 \Bigg] \\
& + \cdots
\end{aligned}
$$

Solving each order of δ separately, we have

(i) Zeroth order equation

$$(43)\quad 0 = \int_0^{2\pi} \left\{ \sum_{n=0}^{\infty} \left[-B_{3n}^{(0)} \frac{\partial G_D}{\partial r_0'} + B_{4n}^{(0)} G_D \right] \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} (n\theta_0')\, b\, d\theta_0' \right\}_{r_0'=b,\, r'=b^+}$$

$$+ \int_0^{2\pi} \left\{ \sum_{n=0}^{\infty} \left[+A_{3n}^{(0)} \frac{\partial G_D^{(0)}}{\partial r_0} - A_{4n}^{(0)} G_D^{(0)} \right] \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} (n\theta_0)\, a\, d\theta_0 \right\}_{r_0=a,\, r'=b^+}$$

(ii) First order equation

$$(44)\quad \int_0^{2\pi} \left\{ \sum_{n=0}^{\infty} \left[-A_{3n}^{(0)} \frac{\partial G_D^{(1)}}{\partial r_0} + A_{4n}^{(0)} G_D^{(1)} \right] \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} (n\theta_0)\, a\, d\theta_0 \right\}_{r_0=a,\, r'=b^+}$$

$$= \int_0^{2\pi} \left\{ \sum_{n=0}^{\infty} \left[-B_{3n}^{(1)} \frac{\partial G_D}{\partial r_0} + B_{4n}^{(1)} G_D \right] \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} (n\theta_0')\, b\, d\theta_0' \right\}_{r_0'=b,\, r'=b^+}$$

$$+ \int_0^{2\pi} \left\{ \sum_{n=0}^{\infty} \left[A_{3n}^{(1)} \frac{\partial G_D^{(0)}}{\partial r_0} - A_{4n}^{(1)} G_D^{(0)} \right] \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} (n\theta_0)\, a\, d\theta_0 \right\}_{r_0=a,\, r'=b^+}$$

We see that the first order solution is coupled to the zeroth order solution. Furthermore, the form of $G_D^{(1)}$ is such that an j mode zeroth order term will produce (j+1) and (j-1) mode first order terms. A similar equation holds on ψ .

The inner boundary integral equations are handled in the same manner with r' equal to a^-, i.e., the field point just short of the inner boundary. For example, we now have

$$(45)\quad 0 = \int_0^{2\pi} \left\{ -\Phi(b) \frac{\partial G_D}{\partial r_0'} + G_D \frac{\partial \Phi(b)}{\partial r_0'} \right\}_{r_0'=b,\, r=a^-} b\, d\theta_0'$$

$$+ \int_0^{2\pi} \left\{ \Phi(a) \frac{\partial G_D}{\partial r_0} - G_D \frac{\partial \Phi(a)}{\partial r_0} \right\}_{r_0=a,\, r=a^-} a\, d\theta_0$$

Now the first integral is a mixed type. Using the same expansions as before we have

(i) Zeroth order equation

$$(46)\quad 0 = \int_0^{2\pi} \left\{ \sum_{n=0}^{\infty} \left[-B_{3n}^{(0)} \frac{\partial G_D^{(0)}}{\partial r_0'} + B_{4n}^{(0)} G_D^{(0)} \right] \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (n\theta_0') \right\}_{r_0'=b,\, r=a^-} b\, d\theta_0'$$

$$+ \int_0^{2\pi} \left\{ \sum_{n=0}^{\infty} \left[A_{3n}^{(0)} \frac{\partial G_D}{\partial r_0} - A_{4n}^{(0)} G_D \right] \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (n\theta_0) \right\}_{r_0=a,\, r=a^-} a\, d\theta_0$$

(ii) First order equations

$$(47)\quad 0 = \int_0^{2\pi} \left\{ \sum_{n=0}^{\infty} \left[-B_{3n}^{(0)} \frac{\partial G_D^{(1)}}{\partial r_0'} + B_{4n}^{(0)} G_D^{(1)} - B_{3n}^{(1)} \frac{\partial G_D^{(0)}}{\partial r_0'} + B_{4n}^{(1)} G_D^{(0)} \right] \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (n\theta_0') \right\}_{r_0'=b,\, r=a^-} b\, d\theta$$

$$+ \int_0^{2\pi} \left\{ \sum_{n=0}^{\infty} \left[A_{3n}^{(1)} \frac{\partial G_D}{\partial r_0} - A_{4n}^{(1)} G_D \right] \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (n\theta_0) \right\}_{r_0=a,\, r=a^-} a\, d\theta_0$$

with a similar equation on ψ, etc.

We now choose $p(a)$ to represent a single mode, i.e., $A_{1n} = 0$, $n \neq j$ and $A_{1j} = P_j$. The zeroth order boundary integral equations and boundary conditions will also only involve this one mode. The first order boundary integral equations will be satisfied only with the $(j+1)$ and $(j-1)$ modes; the boundary conditions will therefore be applied only for these modes as well.

D. ZEROTH ORDER SOLUTION

The boundary conditions, equations (15-17), apply for each order separately and may be used directly as may the acoustic fluid boundary integral equation (30). The outer boundary shell equation on Φ, equation (46), integrates to

$$(48)\quad 0 = \sum_{n=0}^{\infty} \Big[b\left\{ -B_{3n}^{(0)} k_D J_n'(k_D b) + B_{4n}^{(0)} J_n(k_D b) \right\} + a\left\{ A_{3n}^{(0)} k_D J_n'(k_D a) - A_{4n}^{(0)} J_n(k_D a) \right\} \Big] H_n(k_D b) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (n\theta')$$

Since $H_n(k_D b)$ cannot be zero and $\begin{Bmatrix} \sin \\ \cos \end{Bmatrix} n\theta'$ are orthogonal, we have

$$(49)\quad 0 = k_D \left\{ -b\, B_{3n}^{(0)} J_n'(k_D b) + a\, A_{3n}^{(0)} J_n'(k_D a) \right\} + b\, B_{4n}^{(0)} J_n(k_D b) - a\, A_{4n}^{(0)} J_n(k_D a)$$

Similarly for ψ, with $H_n(k_R b)$ not equal to zero;

$$(50)\quad 0 = k_R \left\{ -b\, B_{5n}^{(0)} J_n'(k_R b) + a\, A_{6n}^{(0)} J_n'(k_R a) \right\} + b\, B_{6n}^{(0)} J_n(k_R b) - a\, A_{6n}^{(0)} J_n(k_R b)$$

The inner boundary shell equations give similar results provided $J_n(k_D a)$ and $J_n(k_R a)$ are not zero.

$$(51)\quad 0 = k_D \left\{ -b\, B_{3n}^{(0)} H_n'(k_D b) + a\, A_{3n}^{(0)} H_n'(k_D a) \right\} + b\, B_{4n}^{(0)} H_n(k_D b) - a\, A_{4n}^{(0)} H_n(k_D a)$$

$$(52)\quad 0 = k_R \left\{ -b\, B_{5n}^{(0)} H_n'(k_R b) + a\, A_{5n}^{(0)} H_n'(k_R a) \right\} + b\, B_{6n}^{(0)} H_n(k_R b) - a\, A_{6n}^{(0)} H_n(k_R a)$$

In summary, our equations are

(53) (a) $-A_{1n} = (-\rho_s\omega^2 + 2\mu n^2/a^2)\, A_{3n} + 2\mu\left[-A_{4n}/a + n A_{5n}/a^2 - n A_{6n}/a\right]$

(b) $0 = -n\mu A_{3n}/a^2 + n\mu A_{4n}/a^2 + \left[\frac{\rho_s\omega^2}{2} - \mu n^2/a^2\right] A_{5n} + \mu A_{6n}/a$

(c) $0 = B_{1n} + \left[-\rho_s\omega^2 + 2\mu n^2/b^2\right] B_{3n} + 2\mu\left[-B_{4n}/b + n B_{5n}/b^2 - n B_{6n}/b\right]$

(d) $0 = -n\mu B_{3n}/b^2 + \mu n B_{4n}/b + \left[\frac{\rho_s\omega^2}{2} - \mu n^2/b^2\right] B_{5n} + \mu B_{6n}/b$

(e) $0 = -B_{2n} + \rho_F\omega^2\left[B_{4n} - n B_{5n}/b\right]$

(f) $0 = -B_{2n} + k H_n'(kb)\, B_{1n} / H_n(kb)$

(g) $0 = k_D\left[-b J_n'(k_D b)\, B_{3n} + a J_n'(k_D a)\, A_{3n}\right] + b J_n(k_D b)\, B_{4n} - a J_n(k_D a)\, A_{4n}$

(h) $0 = k_R\left[-b J_n'(k_R b)\, B_{5n} + a J_n'(k_R a)\, A_{5n}\right] + b J_n(k_R b)\, B_{6n} - a J_n(k_R a)\, A_{6n}$

(i) $0 = k_D\left[-b Y_n'(k_D b)\, B_{3n} + a Y_n'(k_D a)\, A_{3n}\right] + b Y_n(k_D b)\, B_{4n} - a Y_n(k_D a)\, A_{4n}$

(j) $0 = k_R\left[-b Y_n'(k_R b)\, B_{5n} + a Y_n'(k_R a)\, A_{5n}\right] + b Y_n(k_R b)\, B_{6n} - a Y_n(k_R a)\, A_{6n}$

We have dropped the superscript (0) from the coefficients in these equations and trivially combined the last four equations to give real rather than complex expressions, although A_{in}, B_{in} are complex anyway due to the complex term in the sixth equation. If A_{1n} is zero for $n \neq j$, all coefficients are zero for $n \neq j$. This then leaves only one mode, $n = j$, to be determined. These equations may readily be checked by the well known solution for the concentric case; they are solved here to indicate the procedure to be used for the higher order terms which have the <u>same</u> coefficients but a modified forcing function (inhomogeneous terms).

We may nondimensionalize these equations using a reference "potential", Φ_j, related to the reference pressure, P_j by $P_j = \omega^2 \rho_s \Phi_j = (\lambda+2u) k_D{}^2 \Phi_j$

We use:

(54) $$\alpha_D = k_D a,\quad \alpha_R = k_R a,\quad \beta_D = k_D b,\quad \beta_R = k_R b,\quad \gamma = k b$$
$$e = \rho_F/\rho_S,\quad \tau = 2\mu/(\lambda+2\mu) = 2\alpha_D^2/\alpha_R^2,\quad h_j = \left[\gamma H_j'(\gamma)/H_j(\gamma)\right]^{-1}$$

and

(55)
$$\begin{aligned}
\tilde{A}_1 &= A_{1j}\, a^2/(\lambda+2\mu)\Phi_j = \alpha_D^2 &;\quad \tilde{B}_1 &= \left(\tfrac{\rho_S}{\rho_F}\right) B_{1j}\, b^2/(\lambda+2\mu)\Phi_j \\
& &\quad \tilde{B}_2 &= \left(\tfrac{\rho_S}{\rho_F}\right) B_{2j}\, b^3/(\lambda+2\mu)\Phi_j \\
\tilde{A}_3 &= A_{3j}/\Phi_j &;\quad \tilde{B}_3 &= B_{3j}/\Phi_j \\
\tilde{A}_4 &= A_{4j}\cdot a/\Phi_j &;\quad \tilde{B}_4 &= B_{4j}\cdot b/\Phi_j \\
\tilde{A}_5 &= A_{5j}/\Phi_j &;\quad \tilde{B}_5 &= B_{5j}/\Phi_j \\
\tilde{A}_6 &= A_{6j}\cdot a/\Phi_j &;\quad \tilde{B}_6 &= B_{6j}\cdot b/\Phi_j
\end{aligned}$$

leading to

(56) (a) $-\alpha_D^2 = \left[\tau j^2 - \alpha_D^2\right]\tilde{A}_3 + \tau\left[-\tilde{A}_4 + j\tilde{A}_5 - j\tilde{A}_6\right]$

(b) $0 = -j\tau\tilde{A}_3 + j\tau\tilde{A}_4 + \left[\alpha_D^2 - j^2\tau\right]\tilde{A}_5 + \tau\tilde{A}_6$

(c) $0 = e\tilde{B}_1 + \left[j^2\tau - \beta_D^2\right]\tilde{B}_3 + \tau\left[-\tilde{B}_4 + j\tilde{B}_5 - j\tilde{B}_6\right]$

(d) $0 = -j\tau\tilde{B}_3 + j\tau\tilde{B}_4 + \left[\beta_D^2 - j^2\tau\right]\tilde{B}_5 + \tau\tilde{B}_6$

(e) $0 = -\tilde{B}_2 + \beta_D^2\tilde{B}_4 - j\beta_D^2\tilde{B}_5$

(f) $0 = -\tilde{B}_2 + \tilde{B}_1/h_j$

(g) $0 = -\beta_D J_j'(\beta_D)\tilde{B}_3 + \alpha_D J_j'(\alpha_D)\tilde{A}_3 + J_j(\beta_D)\tilde{B}_4 - J_j(\alpha_D)\tilde{A}_4$

(h) $0 = -\beta_R J_j'(\beta_R)\tilde{B}_5 + \alpha_R J_j'(\alpha_R)\tilde{B}_5 + J_j(\beta_R)\tilde{B}_6 - J_j(\alpha_R)\tilde{A}_6$

(i) $0 = -\beta_D Y_j'(\beta_D)\tilde{B}_3 + \alpha_D Y_j'(\alpha_D)\tilde{A}_3 + Y_j(\beta_D)\tilde{B}_4 - Y_j(\alpha_D)\tilde{A}_4$

(j) $0 = -\beta_R Y_j'(\beta_R)\tilde{B}_5 + \alpha_R Y_j'(\alpha_R)\tilde{A}_5 + Y_j(\beta_R)\tilde{B}_6 - Y_j(\alpha_R)\tilde{A}_6$

Although this may appear to be a formidable system - ten equations in ten complex unknowns corresponding to a twenty by twenty matrix system - the equations are readily reduced, eliminating variables until a simpler system is reached, e.g., B_1 and B_2 are eliminated directly using the fifth and sixth equations:

$$\text{(a)} \quad \tilde{B}_2 = \beta_D^2 \left[\tilde{B}_4 - j \tilde{B}_5 \right]$$

(57)

$$\text{(b)} \quad \tilde{B}_1 = h \beta_D^2 \left[\tilde{B}_4 - j \tilde{B}_5 \right]$$

A3, A5 can be written in terms of A4 and A6 with similar equations for the B3, B5

$$\text{(a)} \quad \tilde{A}_3 = K_3 + C_{34} \tilde{A}_4 + C_{36} \tilde{A}_6$$

$$\text{(b)} \quad \tilde{A}_5 = K_5 + C_{54} \tilde{A}_4 + C_{56} \tilde{A}_6$$

(58)

$$\text{(c)} \quad \tilde{B}_3 = D_{34} B_4 + D_{36} B_6$$

$$\text{(d)} \quad \tilde{B}_5 = D_{54} B_4 + D_{56} B_6$$

using the first four equations,

$$K_3 = \alpha_D^2 (\alpha_D^2 - \tau j^2) / [(\alpha_D^2 - \tau j^2)^2 - \tau^2 j^2]$$

with

$$K_5 = j \tau \alpha_D^2 / [\quad " \quad]$$

$$C_{34} = C_{56} = - \tau \alpha_D^2 / [\quad " \quad]$$

$$C_{36} = C_{54} = (-j\tau^2 - j\tau\alpha_D^2 + j^3\tau) / [\quad " \quad]$$

and

$$D_{34} = - \beta_D^2 (\tau - e h \beta_D^2) / [(\beta_D^2 - \tau j^2)^2 - j\tau(j\tau - e h j \beta_D^2)]$$

$$D_{36} = D_{54} = [j\tau (j^2\tau - \beta_D^2) - \tau(j\tau - e h j \beta_D^2)] / [\quad " \quad]$$

$$D_{56} = - \beta_D^2 \tau / [\quad " \quad]$$

The last four equations (56 g-j) are rewritten as

(60)
$$
\begin{aligned}
\tilde{B}_3 &= \frac{r_D}{\beta_D s_D}\,\tilde{B}_4 + \frac{2}{\pi \alpha_D \beta_D s_D}\,\tilde{A}_4 \\
\tilde{A}_3 &= \frac{-2}{\pi \alpha_D \beta_D s_D}\,\tilde{B}_4 + \frac{q_D}{\alpha_D s_D}\,\tilde{A}_4 \\
\tilde{B}_5 &= \frac{r_R}{\beta_R s_R}\,\tilde{B}_6 + \frac{2}{\pi \alpha_R \beta_R s_R}\,\tilde{A}_6 \\
\tilde{A}_5 &= \frac{-2}{\pi \alpha_R \beta_R s_R}\,\tilde{B}_6 + \frac{q_R}{\alpha_R s_R}\,\tilde{A}_6
\end{aligned}
$$

where

$$
\begin{aligned}
p(a,b) &= J_j(a)\,Y_j(b) - J_j(b)\,Y_j(a) \\
q(a,b) &= J_j(a)\,Y_j'(b) - J_j'(b)\,Y_j(a) \\
r(a,b) &= J_j'(a)\,Y_j(b) - J_j(b)\,Y_j'(a) \\
s(a,b) &= J_j''(a)\,Y_j'(b) - J_j'(b)\,Y_j'(a)
\end{aligned}
$$

and $p_D = p(\alpha_D, \beta_D)$ etc. We then have a four by four system. Using $K_D = 2/(\pi\alpha_D\beta_D S_D)$ and $K_R = 2/(\pi\alpha_R\beta_R S_R)$ we have,

(61)
$$
\begin{bmatrix}
K_D & 0 & \left[r_D/\beta_D s_D - D_{34}\right] & -D_{36} \\
0 & K_R & -D_{54} & \left[r_R/\beta_R s_R - D_{56}\right] \\
\left[q_D/\alpha_D s_D - C_{34}\right] & -C_{36} & -K_D & 0 \\
-C_{54} & \left[q_R/\alpha_R s_R - C_{56}\right] & 0 & -K_R
\end{bmatrix}
\begin{bmatrix}
\tilde{A}_4 \\ \tilde{A}_6 \\ \tilde{B}_4 \\ \tilde{B}_6
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ K_3 \\ K_5
\end{bmatrix}
$$

with a solution given explicitly in appendix A.

By backwards substitution, we can evaluate all of the unknown coefficients. These must, of course, agree with the well-known solution for the concentric case and are carried out in detail to serve as a procedural check for the higher order solutions.

E. FIRST ORDER SOLUTION

Again, we use the boundary conditions and acoustic fluid boundary integral equation directly for the first order coefficients. The outer boundary shell integral equation on Φ, equation (44), integrates to

$$(62)\quad 0=\sum_{n=0}^{\infty}\left\{\left[\left\{-B_{3n}^{(1)}\,b\,k_D\cdot 2\pi\,J_n'(k_Db)+B_{4n}^{(1)}\cdot b\cdot 2\pi\,J_n(k_Db)\right\}H_n(k_Db)\right.\right.$$
$$\left.+\left\{A_{3n}^{(1)}\cdot a\cdot k_D\cdot 2\pi\,J_n'(k_Da)-A_{4n}^{(1)}\cdot a\cdot 2\pi\cdot J_n(k_Da)\right\}H_n(k_Db)\right]$$
$$\left.\cdot\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}(n\theta')\right\}$$
$$+\sum_{n=0}^{\infty}\left\{\left\{A_{3n}^{(0)}\cdot a\cdot\pi\cdot k_D^2\cdot J_n'(k_Da)-A_{4n}^{(0)}\cdot a\cdot\pi\cdot k_D\,J_n(k_Da)\right\}\right.$$
$$\left.\cdot\left[\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}((n-1)\theta')\cdot H_{n-1}(k_Db)-\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}((n+1)\theta')\cdot H_{n+1}(k_Db)\right]\right\}$$

and a similar equation on ψ with A_{5n}, A_{6n}, B_{5n}, B_{6n} and k_R replacing A_{3n}, A_{4n}, B_{3n}, B_{4n} and k_D.

The inner boundary shell integral equation on Φ becomes

$$(63)\quad 0=\sum_{n=0}^{\infty}\left[\left\{-B_{3n}^{(0)}\cdot b\cdot k_D^2\cdot\pi\,H_n'(k_Db)+B_{4n}^{(0)}\cdot b\cdot k_D\cdot\pi\,H_n(k_Db)\right\}\right.$$
$$\left.\cdot\left\{\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}[(n+1)\theta]\cdot J_{n+1}(k_Da)-\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}[(n-1)\theta]\,J_{n-1}(k_Da)\right\}\right]+$$

$$+ \sum_{n=0}^{\infty} \Big[\Big\{ - B_{3n}^{(1)} \cdot b \cdot 2\pi \cdot k_D \, H_n'(k_D b) + B_{4n}^{(1)} \cdot b \cdot 2\pi \cdot H_n(k_D b)$$

$$+ A_{3n}^{(1)} \cdot a \cdot 2\pi \cdot k_D \cdot H_n'(k_D a) - A_{4n}^{(1)} \cdot a \cdot 2\pi \cdot H_n(k_D a) \Big\}$$

$$\cdot \Big\{ J_n(k_D a) \cdot \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} (n\Theta) \Big\} \Big]$$

with again a similar equation on ψ .

Clearly if there is only one zeroth order mode, $n = j$, there are only two first order modes, $n = j+1$ and $n = j-1$. Consider the $j+1$ mode first. The governing equations are

(64) $$- B_{3\,j+1}^{(1)} \cdot k_D b \cdot J_{j+1}'(k_D b) + B_{4\,j+1}^{(1)} \cdot b \cdot J_{j+1}(k_D b)$$

$$+ A_{3\,j+1}^{(1)} \cdot k_D a \cdot J_{j+1}'(k_D a) - A_{4\,j+1}^{(1)} \cdot a \cdot J_{j+1}(k_D a)$$

$$= \frac{1}{2} \left[A_{3j}^{(0)} \cdot k_D^2 a \cdot J_j'(k_D a) - A_{4j}^{(0)} \cdot k_D a \cdot J_j(k_D a) \right]$$

$$= \frac{1}{2} \left[B_{3j}^{(0)} \cdot k_D^2 b \, J_j'(k_D b) - B_{4j}^{(0)} \cdot k_D b \cdot J_j(k_D b) \right]$$

and

$$- B_{3\,j+1}^{(1)} \cdot k_D b \cdot H_{j+1}'(k_D b) + B_{4\,j+1}^{(1)} \cdot b \cdot H_{j+1}(k_D b)$$

$$+ A_{3\,j+1}^{(1)} \cdot k_D a \cdot H_{j+1}'(k_D a) - A_{4\,j+1}^{(1)} \cdot a \cdot H_{j+1}(k_D a)$$

(65)

$$= \frac{1}{2} \left[B_{3j}^{(0)} \cdot k_D^2 b \cdot H_j'(k_D b) - B_{4j}^{(0)} \cdot k_D b \cdot H_j(k_D b) \right]$$

and two similar equations on ψ . The j-1 mode is essentially the same with a sign change for the inhomogeneous (zeroth order) terms

$$(66)\quad \begin{aligned} &- B_{3\,j-1}^{(1)}\, k_D b\, J_{j-1}'(k_D b) + B_{4\,j-1}^{(1)} \cdot b \cdot J_{j-1}(k_D b) \\ &+ A_{3\,j-1}^{(1)}\, k_D a\, J_{j-1}'(k_D a) - A_{4\,j-1}^{(1)} \cdot a \cdot J_{j-1}(k_D a) \\ &= \frac{-1}{2}\left[A_{3j}^{(0)}\, k_D^2 a\, J_j'(k_D a) - A_{4j}^{(0)} \cdot k_D a \cdot J_j(k_D a)\right] \end{aligned}$$

and

$$(67)\quad \begin{aligned} &- B_{3\,j-1}^{(1)} \cdot k_D b \cdot H_{j-1}'(k_D b) + B_{4\,j-1}^{(1)} \cdot b \cdot H_{j-1}(k_D b) \\ &+ A_{3\,j-1}^{(1)}\, k_D a\, H_{j-1}'(k_D a) - A_{4\,j-1}^{(1)} \cdot a \cdot H_{j-1}(k_D a) \\ &= \frac{-1}{2}\left[B_{3j}^{(0)}\, k_D^2 b\, H_j'(k_D b) - B_{4j}^{(0)} \cdot k_D b \cdot H_j(k_D b)\right] \end{aligned}$$

and two similar equations on ψ .

We note that the homogeneous part of these equations plus the first order boundary conditions and the acoustic boundary integral equation are identical to the zeroth order equations with j+1 and j-1 for the first order coefficients replacing j for the zeroth order coefficients. The inhomogeneous terms are modified and now appear only in the four shell boundary integral equations. Then we can reuse the solution scheme for the zeroth order j mode coefficients to solve for the first order (j+1) and (j-1) mode coefficients with a minor change. We again may write equations (65) and (67) with real coefficients by combining with the previous equations - equivalently, we replace Hn by Yn in these equations.

We may again nondimensionalize using an additional length scale factor of a for these first order terms which must, in final form, be multiplied by the length δ in order to be compared to the zeroth order terms; e.g.,

$$\tilde{A}^{(1)}_{3\,j+1} = A^{(1)}_{3\,j+1} \cdot a/\Phi_j$$
$$\tilde{A}^{(1)}_{4\,j+1} = A^{(1)}_{4\,j+1} \cdot a^2/\Phi_j$$

$$\tilde{B}^{(1)}_{3\,j+1} = B^{(1)}_{3\,j+1} \cdot a/\Phi_j$$
$$\tilde{B}^{(1)}_{4\,j+1} = B^{(1)}_{4\,j+1} \cdot ab/\Phi_j$$

etc.

The same scale factor is used for both $A^{(1)}$ and $B^{(1)}$ to keep the same form of equations as in the zeroth order case. We then have

(a) $$0 = [\tau(j+1)^2 - \alpha_0^2]\,\tilde{A}^{(1)}_{3\,j+1} + \tau[-\tilde{A}^{(1)}_{4\,j+1} + (j+1)\,\tilde{A}^{(1)}_{5\,j+1} - (j+1)\,\tilde{A}^{(1)}_{6\,j+1}]$$

(b) $$0 = -\tau(j+1)\,\tilde{A}^{(1)}_{3\,j+1} + \tau(j+1)\,\tilde{A}^{(1)}_{4\,j+1} + [\alpha_0^2 - (j+1)^2\tau]\,\tilde{A}^{(1)}_{5\,j+1} + \tau\,\tilde{A}^{(1)}_{6\,j+1}$$

(c) $$0 = \tilde{B}^{(1)}_{1\,j+1} + [\tau(j+1)^2 - \beta_0^2]\,\tilde{B}^{(1)}_{3\,j+1} + \tau[-\tilde{B}^{(1)}_{4\,j+1} + (j+1)\,\tilde{B}^{(1)}_{5\,j+1} - (j+1)\,\tilde{B}^{(1)}_{6\,j+1})$$

(68)

(d) $$0 = -\tau(j+1)\,\tilde{B}^{(1)}_{3\,j+1} + \tau(j+1)\,\tilde{B}^{(1)}_{4\,j+1} + [\beta_0^2 - (j+1)^2\tau]\,\tilde{B}^{(1)}_{5\,j+1} + \tau\,\tilde{B}^{(1)}_{6\,j+1}$$

(e) $$0 = -\tilde{B}^{(1)}_{2\,j+1} + \beta_0^2\,[\tilde{B}^{(1)}_{4\,j+1} - (j+1)\,\tilde{B}^{(1)}_{5\,j+1}]$$

(f) $$0 = -\tilde{B}^{(1)}_{2\,j+1} + \tilde{B}^{(1)}_{1\,j+1}/h_{j+1}$$

(g) $$L_1 = \frac{\alpha_0}{2}[\beta_0 J_j'(\beta_0)\,\tilde{B}^{(0)}_{3j} - \tilde{B}^{(0)}_{4j} \cdot J_j(\beta_0)]$$

$$= -\tilde{B}^{(1)}_{3\,j+1}\,\beta_0 J_{j+1}'(\beta_0) + \tilde{A}^{(1)}_{3\,j+1}\,\alpha_0 J_{j+1}'(\alpha_0) + \tilde{B}^{(1)}_{4\,j+1}\,J_{j+1}(\beta_0) - \tilde{A}^{(1)}_{4\,j+1}\,J_{j+1}(\alpha$$

(h)
$$L_2 = \frac{\alpha_R}{2}\left[\tilde{B}^{(0)}_{5j}\,\beta_R J_j'(\beta_R) - \tilde{B}^{(0)}_{6j} J_j(\beta_R)\right]$$

$$= -\tilde{B}^{(1)}_{5j+1}\,\beta_R J_{j+1}'(\beta_R) + \tilde{A}^{(1)}_{5j+1}\,\alpha_R J_{j+1}'(\alpha_R) + \tilde{B}^{(1)}_{6j+1} J_{j+1}(\beta_R) - \tilde{A}^{(1)}_{6j+1} J_{j+1}(\alpha_R)$$

(i)
$$L_3 = \frac{\alpha_D}{2}\left[\tilde{B}^{(0)}_{3j}\,\beta_D Y_j'(\beta_D) - \tilde{B}^{(0)}_{4j} Y_j(\beta_D)\right]$$

$$= -\tilde{B}^{(1)}_{3j+1}\,\beta_D Y_{j+1}'(\beta_D) + \tilde{A}^{(1)}_{3j+1}\,\alpha_D Y_{j+1}'(\alpha_D) + \tilde{B}^{(1)}_{4j+1} Y_{j+1}(\beta_D) - \tilde{A}^{(1)}_{4j+1} Y_{j+1}(\alpha_D)$$

(j)
$$L_4 = \frac{\alpha_R}{2}\left[\tilde{B}^{(0)}_{5j}\,\beta_R Y_j'(\beta_R) - \tilde{B}^{(0)}_{6j} Y_j(\beta_R)\right]$$

$$= -\tilde{B}^{(1)}_{5j+1}\,\beta_R Y_{j+1}'(\beta_R) + \tilde{A}^{(1)}_{5j+1}\,\alpha_R Y_{j+1}'(\alpha_R) + \tilde{B}^{(1)}_{6j+1} Y_{j+1}(\beta_R) - \tilde{A}^{(1)}_{6j+1} Y_{j+1}(\alpha_R)$$

and a similar set of equations with $j-1$ replacing $j+1$ and $-L_1$, $-L_2$, $-L_3$, $-L_4$ replacing L_1, L_2, L_3, L_4.

We see at this point that we have two separate ten by ten systems of complex algebraic equations - each very similar to those of the zeroth order case. We again may reduce the unknowns to yield a feasible arithmetic problem - consider the $j+1$ system as an example:

(69)

(a) $$\tilde{B}^{(1)}_{2j+1} = \beta_D^2\left[\tilde{B}^{(1)}_{4j+1} - (j+1)\quad \tilde{B}^{(1)}_{5j+1}\right]$$

(b) $$\tilde{B}^{(1)}_{1j+1} = h\,\beta_D^2\left[\tilde{B}^{(1)}_{4j+1} - (j+1)\,\tilde{B}^{(1)}_{5j+1}\right]$$

(c) $$\tilde{A}^{(1)}_{3j+1} = C_{34}\,\tilde{A}^{(1)}_{4j+1} + C_{36}\,\tilde{A}^{(1)}_{6j+1}$$

(d) $$\tilde{A}^{(1)}_{5j+1} = C_{54}\,A^{(1)}_{4j+1} + C_{56}\,\tilde{A}^{(1)}_{6j+1}$$

(e) $$\tilde{B}^{(1)}_{3j+1} = D_{34}\,\tilde{B}^{(1)}_{4j+1} + D_{36}\,\tilde{B}^{(1)}_{6j+1}$$

(f) $$\tilde{B}^{(1)}_{5j+1} = D_{54}\,\tilde{B}^{(1)}_{4j+1} + D_{56}\,\tilde{B}^{(1)}_{6j+1}$$

where the C's and D's are the same as before with j+1 replacing n . Clearly the K_3 and K_5 are no longer present since the inhomogeneity in the system has moved to the last four equations.

Those last equations are now

(70)

$$\text{(a)}\quad K_D \tilde{A}^{(1)}_{4j+1} + \left(\frac{r_D}{\Delta_D \beta_D} - D_{34}\right) \tilde{B}^{(1)}_{4j+1} - D_{36} \tilde{B}^{(1)}_{6j+1} = \frac{L_3 J'_{j+1}(\alpha_D) - L_1 Y'_{j+1}(\alpha_D)}{\beta_D \Delta_D} = R_1$$

$$\text{(b)}\quad K_R \tilde{A}^{(1)}_{6j+1} - D_{54} \tilde{B}^{(1)}_{4j+1} + \left(\frac{r_R}{\Delta_R \beta_R} - D_{56}\right) \tilde{B}^{(1)}_{6j+1} = \frac{L_4 J'_{j+1}(\alpha_R) - L_2 Y'_{j+1}(\alpha_R)}{\beta_R \Delta_R} = R_2$$

$$\text{(c)}\quad \left[\frac{q_D}{\alpha_D \Delta_D} - C_{34}\right] \tilde{A}^{(1)}_{4j+1} - C_{36} \tilde{A}^{(1)}_{6j+1} - K_D \tilde{B}^{(1)}_{4j+1} = \frac{L_3 J'_{j+1}(\beta_D) - L_1 Y'_{j+1}(\beta_D)}{\alpha_D \Delta_D} = R_3$$

$$\text{(d)}\quad - C_{54} \tilde{A}^{(1)}_{4j+1} + \left[\frac{q_R}{\alpha_R \Delta_R} - C_{56}\right] \tilde{A}^{(1)}_{6j+1} - K_R \tilde{B}^{(1)}_{6j+1} = \frac{L_4 J'_{j+1}(\beta_R) - L_2 Y'_{j+1}(\beta_R)}{\alpha_R \Delta_R} = R_4$$

remembering that the C, D, p, q, r and s are all modified to have j+1 replace n .

The equations for the (j-1) mode are identical to the above with the corresponding changes of j-1 for j+1 everywhere and with the sign of the R terms reversed. The solution procedure is the same as for the zeroth order, and is also given in appendix A.

To convert these first order coefficients to a form suitable for comparison to the zeroth order solution, they must be multiplied by (δ/a), which is a small number in order to have the first order be a sufficient correction to the zeroth order solution. Higher orders may be calculated in the same manner -- only a new set of L_i need be calculated since the same solution procedure will hold for higher orders as did for the first order, i.e., the same basic program.

CALCULATIONS AND CONCLUSIONS

The primary application of this solution is to a thin steel shell submerged in water. The parameters used were a = 300 cm, b = 303 cm, c = 146000 cm/sec, c_D = 496000 cm/sec and c_R = 270000 cm/sec with ρ_S = 7.7 gm/cm^3 and ρ_F = 1.0 gm/cm^3. Results for the exterior pressure B_1, are shown in Figures 2 and 3 for the case j = 0 and j = 2 respectively for a unit interior forcing pressure. Results for the first order solutions are given on the same figure as the zeroth order solution, but to a scale 50 times as great. Since the first order terms must be multiplied by δ/a to determine their contribution to the exterior pressure field, the actual exterior pressure will depend on δ. For δ = 0.3 cm, $\delta/a = 10^{-3}$ and the first order contribution is generally negligible. There are regions, however, where the zeroth order solution is minimum, e.g. near 300 rad/sec for j = 2. In this region, the first order contributions may be comparable to the zeroth order, but this is for a very limited frequency band. Figure 4 shows results for a thick shell with a = 200 cm and the remaining parameters unchanged. In all cases, ω runs from 200 to 9000 rad/sec. Here the first order and zeroth order solutions are plotted on the same scale and are very similar. This implies the first order contribution, when multiplied by δ/a will be negligible for any example in which δ/a is small enough to allow a first order theory to be used.

The general conclusions to be drawn from this work are first that such an approach is applicable to radiating submerged "nonconcentric" elastic shells and second that nearest neighbor modes are excited in the fluid in the higher order solutions for a single mode forcing function.

Both of these conclusions have already been drawn in the acoustic shell case, reference [1]; we have shown here that they are applicable to the physically more meaningful elastic shell case as well, with some additional effort. A specific result based on these two examples is that the influence of nonconcentricity is essentially negligible in most cases where δ/a is small. Since higher order solutions may be computed with the same basic program, extensions to larger values of (δ/a) are clearly possible for thicker shells. Such analytical solutions may be useful both in their own right and even more importantly as checks on general numerical schemes which might not be otherwise properly constructed to include (or recognize) such effects as nearest neighbor modes.

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2. W. Thompson, "Acoustic Radiation from a Spherical Source Embedded Eccentrically Within a Fluid Sphere", J.A.S.A. 54( ), pp. 1694-1707 (1973).

3. R. P. Shaw, "An Integral Equation Approach to Acoustic Radiation and Scattering", Topics in Ocean Engineering, II, edited: C. Bretschneider, Gulf Publishing Co., Houston, Texas, 1970. [Note: The equations for time harmonic case use inward rather than outward normal--this provides an overview of boundary integral equations].

4. A. E. H. Love, A Treatise on the Mathematical Theory of Elasticity, Chap. XIII, Dover Publishing Co., N.Y.C., N. Y. (Fourth Edition) 1944.

5. Love, ref. [4], has an excellent discussion of this except that his definition uses hi as the inverse of the usual definition of metric coefficient. As a result the equations in Chap. 1 have h_i inverted from the h_i^* used in this paper.

APPENDIX A

Define

$QASD = q_D/\alpha_D s_D - C_{34}$

$QASR = q_R/\alpha_R s_R - C_{56}$

$RBSD = r_D/\beta_D s_D - D_{34}$

Then

$$D = \begin{vmatrix} K_D & 0 & RBSD & -D_{36} \\ 0 & K_R & -D_{54} & RBSR \\ QASD & -C36 & -K_D & 0 \\ -C_{54} & QASR & 0 & -K_R \end{vmatrix}$$

and

$$N_4 = \begin{vmatrix} R_1 & 0 & RBSD & -D_{36} \\ R_2 & K_R & -D_{54} & RBSR \\ R_3 & -C_{36} & -K_D & 0 \\ R_4 & QASR & 0 & -K_R \end{vmatrix}$$

$$N_6 = \begin{vmatrix} K_D & R_1 & RBSD & -D_{36} \\ 0 & R_2 & -D_{54} & RBSR \\ QASD & R_3 & -K_D & 0 \\ -C_{54} & R_4 & 0 & -K_R \end{vmatrix}$$

(Zeroth order)

N_4:
N_6: $R_1 = 0,\ R_2 = 0,\ R_3 = K_3,\ R_4 = K_5$

such that

$$\tilde{A}_{4j}^{(o)} = N_4/D$$

$$\tilde{A}_{6j}^{(o)} = N_6/D$$

(First order)

D (j+1) = D with j+1 replacing j in all terms

N_4 (j+1):

R_1, R_2, R_3, R_4 defined in equation 70

N_6 (j+1):

and a similar set for j-1 contribution.

$$\tilde{A}_{3j}^{(o)} = K_3 + C_{34}\,\tilde{A}_{4j}^{(o)} + C_{36}\,\tilde{A}_{6j}^{(o)}$$

$$\tilde{A}_{5j}^{(o)} = K_5 + C_{54}\,\tilde{A}_{4j}^{(o)} + C_{56}\,\tilde{A}_{6j}^{(o)}$$

$$\tilde{B}_{4j}^{(o)} = K_D \quad -K_3 + QASD*\tilde{A}_{4j}^{(o)} - C_{36}\,\tilde{A}_{6j}^{(o)}$$

$$\tilde{B}_{6j}^{(o)} = K_R \quad -K_5 - C_{54}*\tilde{A}_{4j}^{(o)} + QASR*\tilde{A}_{6j}^{(o)}$$

APPENDIX B

A simple test calculation is the "transparent" shell, i.e. $\rho_F = \rho_S$, $\mu = 0$ and $c_D = c$. Since the pressure is applied at the inner surface, $r = a$, we have

$$p(r,\theta) = \frac{P_n H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \cdot \left\{\begin{matrix} \sin \\ \cos \end{matrix}\right\}(n\theta)$$

On an "offset" circle, we have $r' = b$ given by $r = r' + \delta \cos\theta + \ldots$

$$r = b + \delta \cos\theta + \ldots.$$

and

$$p(r'=b,\theta) = \frac{P_n}{H_n^{(1)}(ka)} \left[H_n^{(1)}(kb) + k \cdot \delta \cdot \cos\theta \cdot \left[H_n^{(1)}(kb)\right]' + \cdots \right] \left\{\begin{matrix} \sin \\ \cos \end{matrix}\right\}(n\theta)$$

$$= p_n^{(0)}(b,\theta) + \delta\, p_{n+1}^{(1)}(b) \cdot \left\{\begin{matrix} \sin \\ \cos \end{matrix}\right\}[(n+1)\theta] + \delta\, p_{n-1}^{(1)}(b) \left\{\begin{matrix} \sin \\ \cos \end{matrix}\right\}[(n-1)\theta] + \cdots$$

where

$$p_n^{(0)}(b,\theta) = P_n \cdot \frac{H_n^{(1)}(kb)}{H_n^{(1)}(ka)} \cdot \left\{\begin{matrix} \sin \\ \cos \end{matrix}\right\}(n\theta)$$

$$p_{n+1}^{(1)}(b) = P_n \cdot \frac{k}{2} \cdot \frac{\left[H_n^{(1)}(kb)\right]'}{\left[H_n(ka)\right]} = p_{n-1}^{(1)}(b)$$

If we return to the integral equation approach, a direct substitution of these parameters into the zeroth order equations leads to

$$B_{1n}^{(0)} = P_n \cdot \frac{H_n(\beta)}{H_n(\alpha)}$$

The first order equations lead to

$$B^{(1)}_{1\,n+1} = -\frac{k}{2}\,\frac{H^{(1)}_{n+1}(kb)}{H^{(1)}_{n}(ka)}\,P_n$$

$$B^{(1)}_{1\,n-1} = -\frac{k}{2}\,\frac{H^{(1)}_{n-1}(kb)}{H^{(1)}_{n}(ka)}\,P_n$$

We must bear in mind, however, that these are values for the pressure at the outer surface measured in terms of θ' and not in terms of θ. We need to convert $\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}(n\,\theta.)$. We need to convert $\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}(n\,\theta)$ to $\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}(n\,\theta')$ which modifies our first order results. We have the definition

$$\theta' = \theta + \sin^{-1}\left\{\frac{\delta}{b}\sin\theta\right\}$$

and to a first order

$$\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}(n\theta) = \left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}(n\theta') - \delta\cdot\frac{n}{2}\left[\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}[(n+1)\theta] + \left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}[(n-1)\theta]\right] + \cdot$$

Then

$$\begin{aligned} p\,(r'=b,\theta') = {} & B^{(0)}_{1n}\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}(n\theta') \\ & + \delta B^{(1)}_{1n+1}\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}[(n+1)\theta'] + \delta\cdot B^{(1)}_{1n-1}\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}[(n-1)\theta'] + \cdots \\ = {} & P_n\cdot\frac{H^{(1)}_{n}(kb)}{H^{(1)}_{n}(ka)}\cdot\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}(n\theta) + \frac{\delta\cdot P_n\cdot k}{2H^{(1)}_{n}(ka)}\Big[-H^{(1)}_{n+1}(kb) \\ & + \frac{n}{2b}H^{(1)}_{n}(kb)\Big]\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}[(n+1)\theta] + \frac{\delta P_n k}{2H^{(1)}_{n}(ka)}\left[H^{(1)}_{n+1}(kb) + \frac{n}{2b}H^{(1)}_{n}(kb)\right]\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}[(n-1)\theta] + \cdots \end{aligned}$$

which agrees with the previous solution.

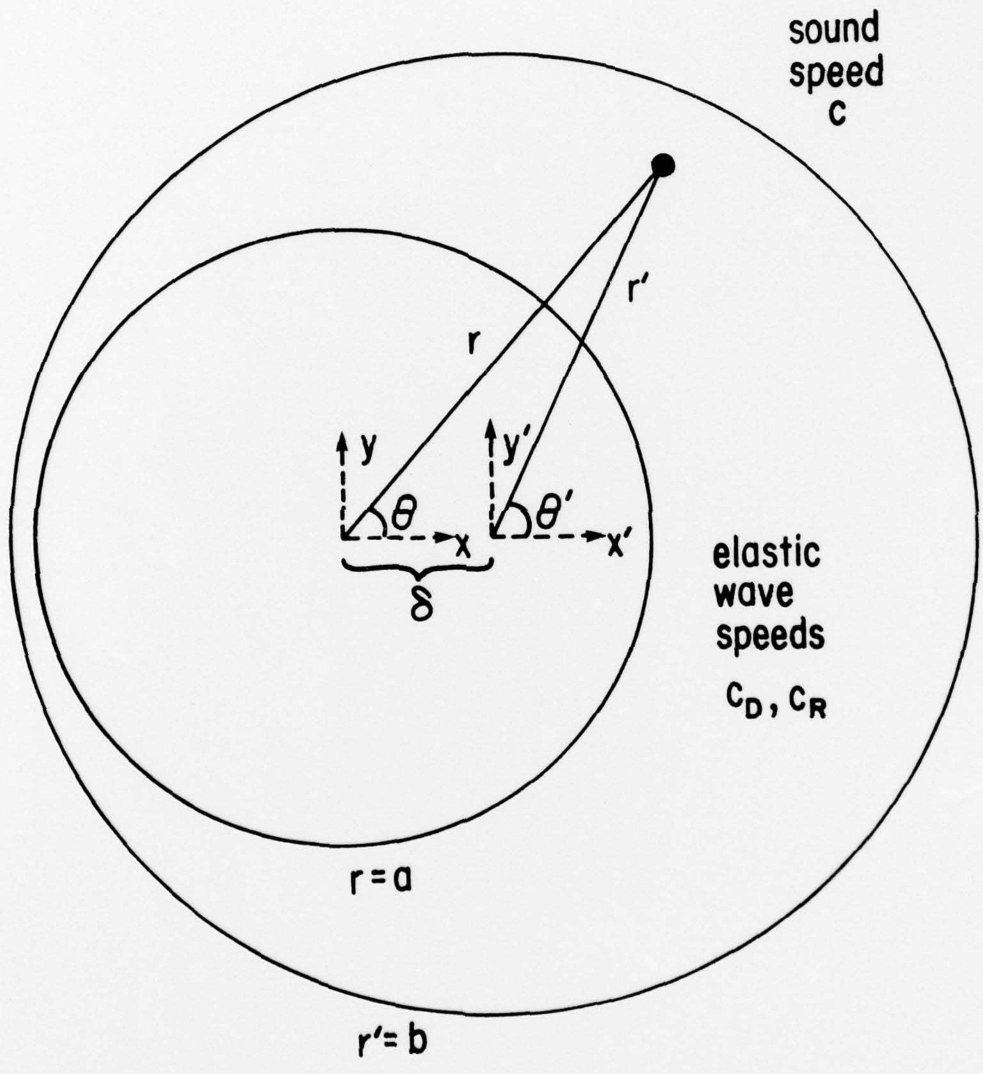

Figure 1. Geometry of Nonconcentric Cylinder

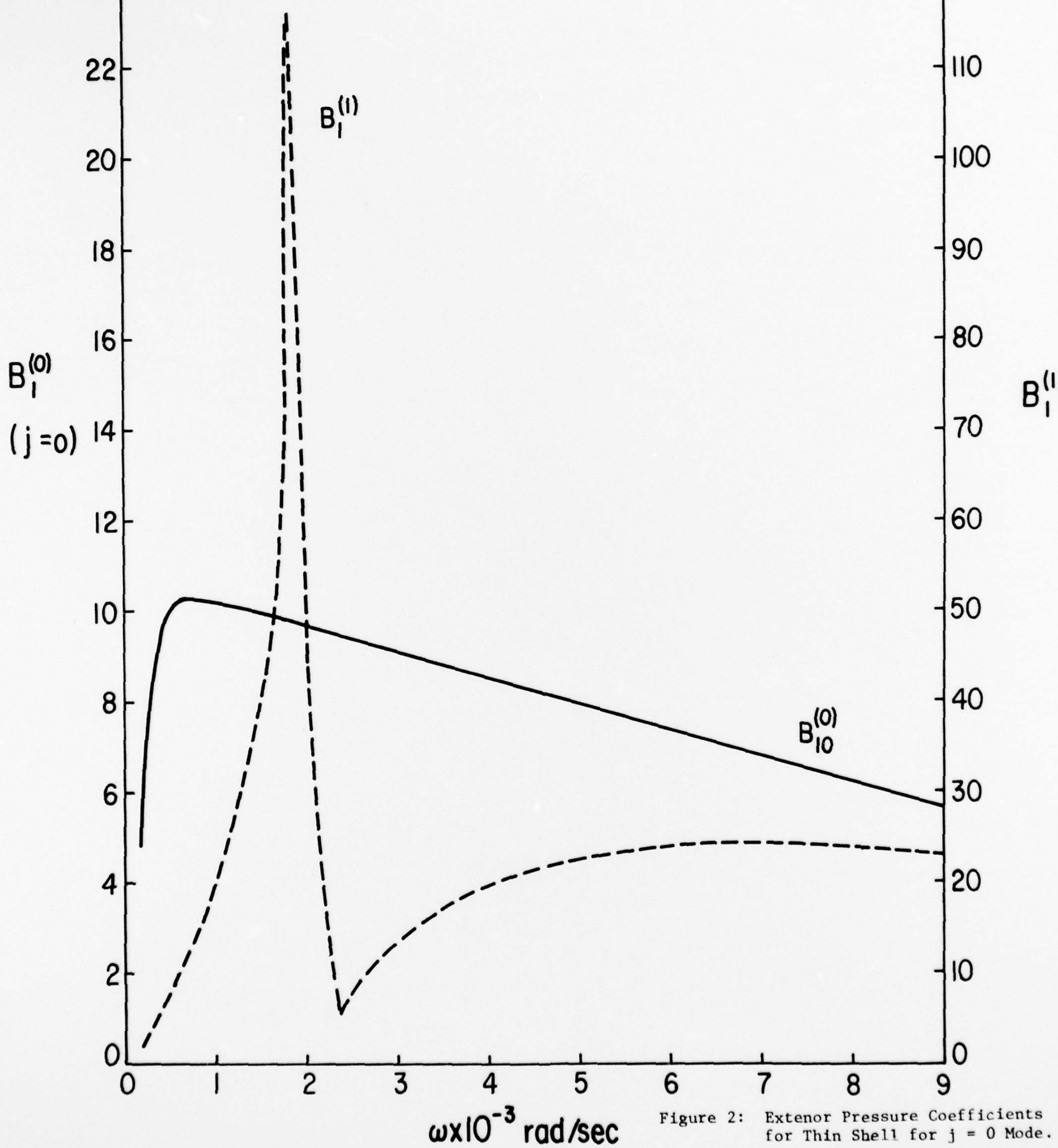

Figure 2: Extenor Pressure Coefficients for Thin Shell for j = 0 Mode.

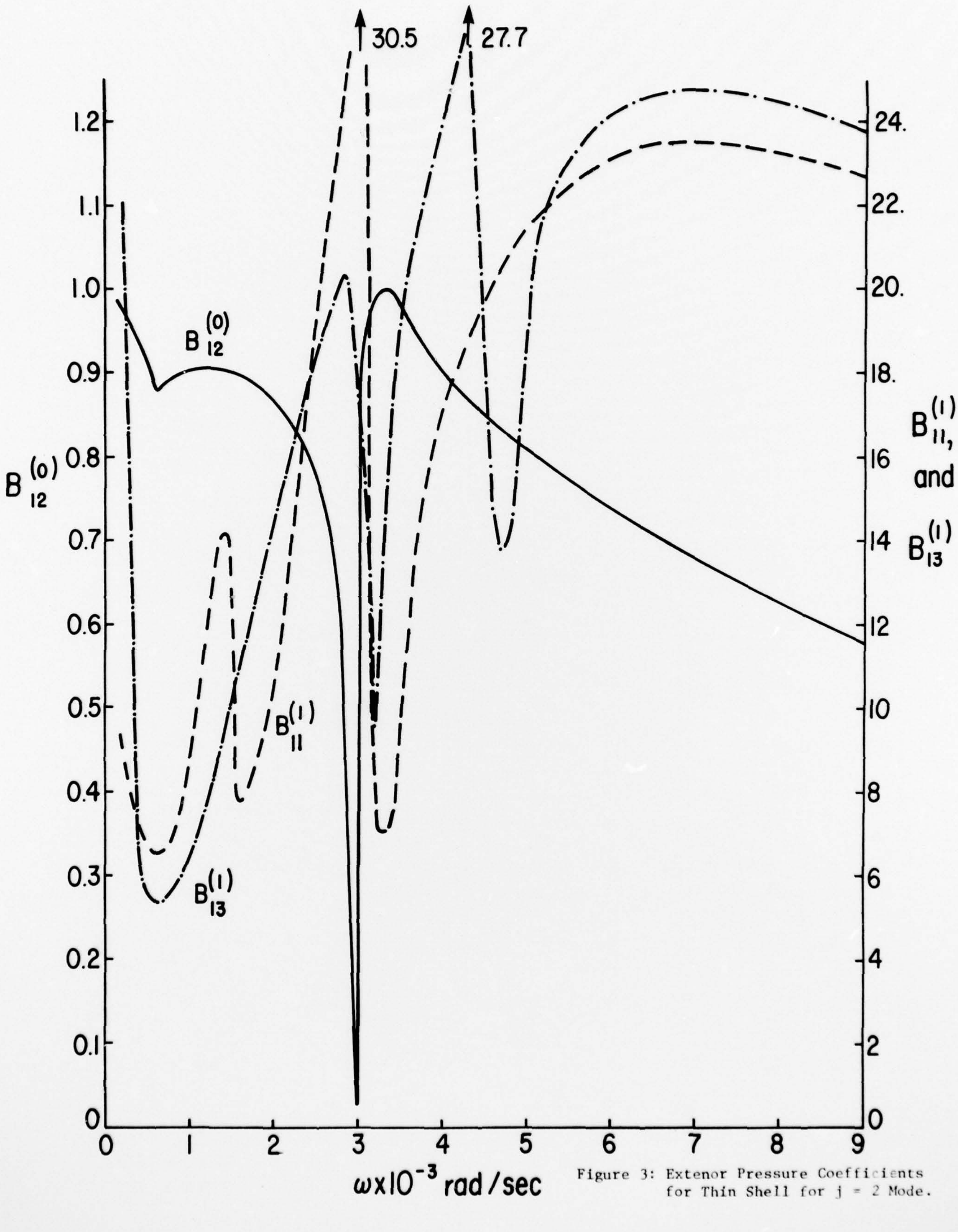

Figure 3: Extenor Pressure Coefficients for Thin Shell for j = 2 Mode.

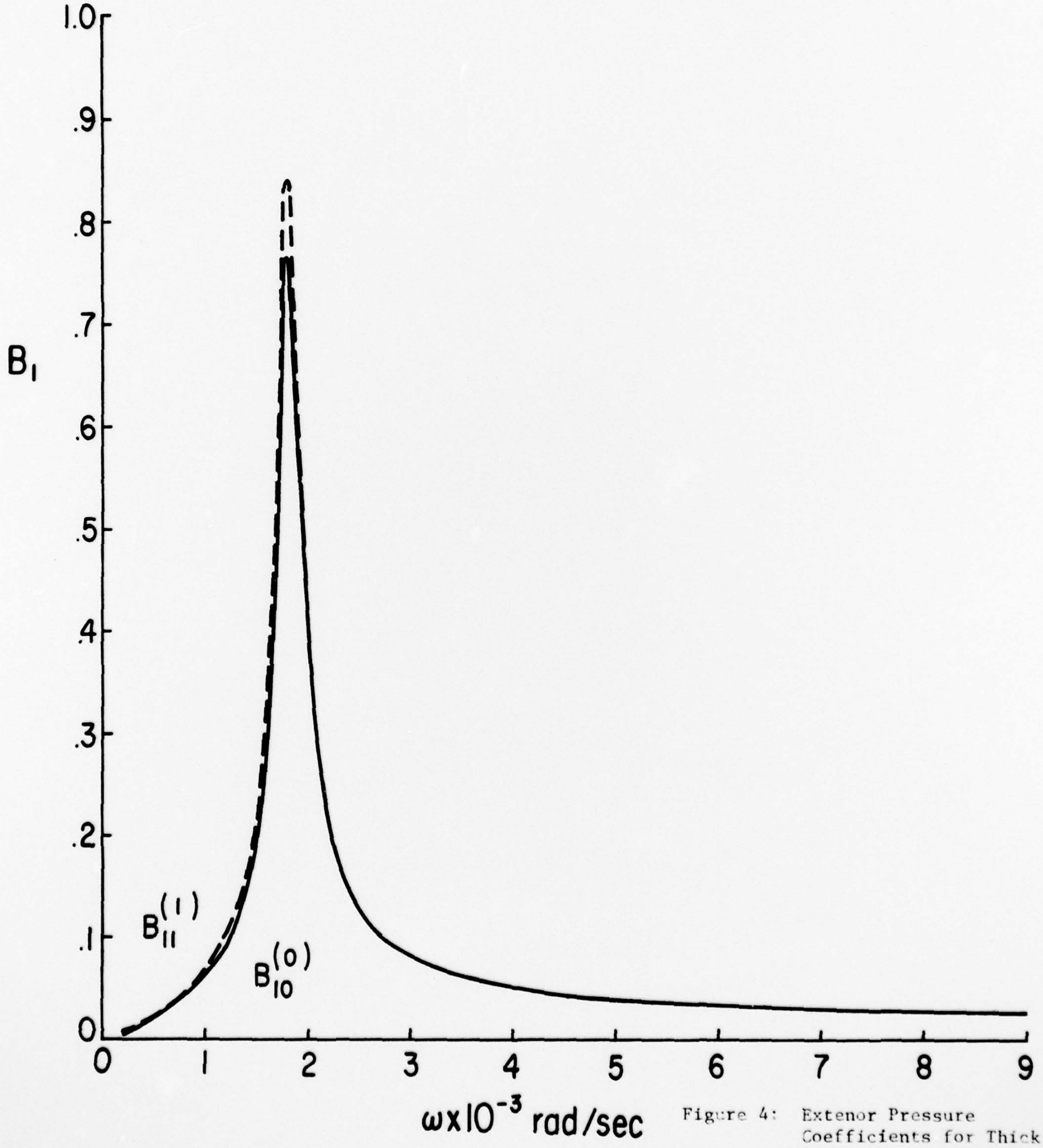

Figure 4: Extenor Pressure Coefficients for Thick Shell for $j = 0$ Mode.

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The acoustic radiation from an elastic cylindrical shell defined by nonconcentric cylindrical surfaces and submerged in an acoustic fluid is studied. The driving mechanism is a time harmonic internal pressure. In particular, the influence of the nonconcentricity parameter, δ, on the resulting radiation field is examined. The solution is obtained by means of an expansion in powers of this parameter about the concentric solution, $\delta=0$. Each order of solution is solved in closed form by means of boundary integral equation methods. Results are given for the zeroth and first order solutions and it is seen that a j^{th} driving mode will introduce nearest neighbor ((j-1) and (j+1)) modes in the first order solution.

delta

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